SMALL COVERS OVER PRISMS

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Abstract. In this paper we calculate the number of equivariant diffeomorphism classes of small covers over a prism.

1. Introduction

The notion of *small covers* is first introduced by Davis and Januszkiewicz [DJ], where a small cover is a smooth closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is a simple convex polytope. This gives a direct connection between equivariant topology and combinatorics. As shown in [DJ], all small covers over a simple convex polytope P^n correspond to all characteristic functions (in this paper we call them $(\mathbb{Z}_2)^n$-colorings) defined on all facets (codimension-one faces) of P^n. However, two small covers over P^n which correspond to different $(\mathbb{Z}_2)^n$-colorings may be equivariantly diffeomorphic. In [LM], Lü and Masuda showed that the equivariant diffeomorphism class of a small cover over P^n agrees with the equivalence class of its corresponding $(\mathbb{Z}_2)^n$-coloring under the action of automorphism group of P^n. Therefore, the number of equivariant diffeomorphism classes of small covers over a certain polytope can be interpreted as that of equivalence classes of $(\mathbb{Z}_2)^n$-colorings, and in particular, the latter one is enumerable. When $n = 2$, a recursive formula for this number is obtained in [LM]. However, there is no instant formula to determine this number if $n > 2$. In [GS], Garrison and Scott used a computer program to enumerate the number of homeomorphism classes of all small covers over a dodecahedron, and the program only needs a few changes if we want to know the number of its equivariant diffeomorphism classes.

Generally, the equivalence classes of $(\mathbb{Z}_2)^n$-colorings of a polytope P^n can be considered as the orbits of all $(\mathbb{Z}_2)^n$-colorings under the action of automorphism group of P^n. Thus, Burnside Lemma (or Cauchy-Frobenius Lemma) can be applied to determine the number of the orbits of all $(\mathbb{Z}_2)^n$-colorings under the action of automorphism group of P^n, which is just the average value of numbers $|\Lambda_g|$, where $|\Lambda_g|$ equals to the number of $(\mathbb{Z}_2)^n$-colorings which are fixed by an automorphism g of P^n. This leads us to give a formula for the number of equivariant diffeomorphism classes of all small covers over a prism (see Theorem 4.1), and in particular, the Euler's totient function is also involved.

The arrangement of this paper is as follows. In Section 2 we review the basic theory about small covers and Burnside Lemma, and calculate the automorphism group of face poset of a prism. In Section 3 we determine the number of all colorings on a

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prism, so that in Section 4 we can obtain a calculation formula of the number of equivariant diffeomorphism classes of all small covers over a prism, and by using a computer program, the first 8 numbers are given.

2. Preliminaries

2.1. **Small covers and colorings.** An n-dimensional convex polytope P^n is said to be *simple* if exactly n faces of codimension one meet at each of its vertices. A closed n-manifold M^n is said to be a *small cover* if it admits an effective $(\mathbb{Z}_2)^n$-action, which is locally isomorphic to the standard action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$, and the orbit space of the action is a simple convex polytope P^n.

Suppose that $\pi : M^n \longrightarrow P^n$ is a small cover over a simple convex polytope P^n. Let $\mathcal{F}(P^n) = \{F_1, ..., F_\ell\}$ be the set of codimension-one faces (facets) of P^n. Then there are ℓ connected submanifolds $M_1, ..., M_\ell$ determined by π and F_i (i.e., $M_i = \pi^{-1}(F_i)$). Each submanifold M_i is fixed pointwise by the $\mathbb{Z}_2$-subgroup G_i of $(\mathbb{Z}_2)^n$, so that each facet F_i corresponds to the $\mathbb{Z}_2$-subgroup G_i. Obviously, such the $\mathbb{Z}_2$-subgroup G_i actually agrees with an element υ_i in $(\mathbb{Z}_2)^n$ as a vector space. For each face F of codimension u, since P^n is simple, there are u facets $F_{i_1}, ..., F_{i_u}$ such that $F = F_{i_1} \cap \cdots \cap F_{i_u}$. Then, the corresponding characteristic submanifolds $M_{i_1}, ..., M_{i_u}$ intersect transversally in the $(n-u)$-dimensional submanifold $\pi^{-1}(F)$, and the isotropy subgroup G_F of $\pi^{-1}(F)$ is a subtorus of rank u and is generated by $G_{i_1}, ..., G_{i_u}$ (or is determined by $\upsilon_{i_1}, ..., \upsilon_{i_u}$ in $(\mathbb{Z}_2)^n$). Thus, this actually gives a characteristic function (see [DJ])

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

defined by $\lambda(F_i) = \upsilon_i$ such that for any face $F = F_{i_1} \cap \cdots \cap F_{i_u}$ of P^n, $\lambda(F_{i_1}), ..., \lambda(F_{i_u})$ are linearly independent in $(\mathbb{Z}_2)^n$. If we regard each nonzero vector of $(\mathbb{Z}_2)^n$ as being a *color*, then the characteristic function λ means that each facet is colored by a color. Thus, we also call λ a $(\mathbb{Z}_2)^n$-*coloring* on P^n here.

Davis and Januszkiewicz [DJ] gave a reconstruction process of M^n by using the $(\mathbb{Z}_2)^n$-coloring λ and the product bundle $(\mathbb{Z}_2)^n \times P^n$ over P^n, so that all small covers over P^n are classified in terms of all $(\mathbb{Z}_2)^n$-colorings on $\mathcal{F}(P^n)$. By $\Lambda(P^n)$ we denote the set of all $(\mathbb{Z}_2)^n$-colorings on P^n. Then we have

Theorem 2.1 (Davis-Januszkiewicz)**.** *Let $\pi : M^n \longrightarrow P^n$ be a small cover over a simple convex polytope P^n. Then all small covers over P^n are given by $\{M(\lambda) | \lambda \in \Lambda(P^n)\}$.*

Remark. Generally speaking, we cannot make sure that there always exist $(\mathbb{Z}_2)^n$-colorings over a simple convex polytope P^n when $n \geq 4$. For example, see [DJ, Nonexamples 1.22]. However, the Four Color Theorem makes sure that every 3-dimensional simple convex polytope always admits $(\mathbb{Z}_2)^3$-colorings.

There is a natural action of $\mathrm{GL}(n, \mathbb{Z}_2)$ on $\Lambda(P^n)$ defined by the correspondence $\lambda \longmapsto \sigma \circ \lambda$, and it is easy to see that such an action on $\Lambda(P^n)$ is free. Without loss of generality, we assume that $F_1, ..., F_n$ of $\mathcal{F}(P^n)$ meet at one vertex p of P^n. Let $e_1, ..., e_n$

be the standard basis of $(\mathbb{Z}_2)^n$. Write $A(P^n) = \{\lambda \in \Lambda(P^n) \big| \lambda(F_i) = e_i, i = 1, ..., n\}$. Then we have

Lemma 2.1. $|\Lambda(P^n)| = |A(P^n)| \times |GL(n, \mathbb{Z}_2)|$.

Note that we know from [AB] that $|GL(n, \mathbb{Z}_2)| = \prod_{k=1}^{n}(2^n - 2^{k-1})$.

2.2. **Automorphisms of polytopes and classification of small covers.** Each point of a simple convex polytope P^n has a neighborhood which is affine isomorphic to an open subset of the positive cone $\mathbb{R}^n_{\geq 0}$, so P^n is an n-dimensional manifold with corners (see [D]). An automorphism of P^n is a self-diffeomorphism of P^n as a manifold with corners, and by $\mathrm{Aut}(P^n)$ we denote the group of automorphisms of P^n. All faces of P^n forms a *poset* (i.e., a partially ordered set by inclusion). An automorphism of $\mathcal{F}(P^n)$ is a bijection from $\mathcal{F}(P^n)$ to itself which preserves the poset structure of all faces of P^n, and by $\mathrm{Aut}(\mathcal{F}(P^n))$ we denote the group of automorphisms of $\mathcal{F}(P^n)$. Each automorphism of $\mathrm{Aut}(P^n)$ naturally induces an automorphism of $\mathcal{F}(P^n)$. It is well-known (see [BP] or [Z]) that two simple convex polytopes are combinatorially equivalent (i.e., their corresponding face posets are isomorphic) if and only if they are diffeomorphic as manifolds with corners. Thus, the natural homomorphism ϕ : $\mathrm{Aut}(P^n) \longrightarrow \mathrm{Aut}(\mathcal{F}(P^n))$ is surjective.

Definition 2.2. Two $(\mathbb{Z}_2)^n$-colorings λ_1 and λ_2 in $\Lambda(P^n)$ are said to be equivalent if there exists an automorphism $h \in \mathrm{Aut}(\mathcal{F}(P^n))$ such that $\lambda_1 = \lambda_2 \circ h$.

Every small cover induces a $(\mathbb{Z}_2)^n$-coloring on its orbit polytope. However, it is possible that two $(\mathbb{Z}_2)^n$-colorings on a simple convex polytope rebuild two equivariantly diffeomorphic small covers. The following theorem shows that the equivalence of $(\mathbb{Z}_2)^n$-colorings exactly determines small covers up to equivariant diffeomorphism.

Theorem 2.3. *Two small covers over a simple convex polytope P^n are equivariantly diffemorphic if and only if their corresponding $(\mathbb{Z}_2)^n$-colorings on P^n are equivalent.*

Readers may find the proof of Theorem 2.3 in [LM], which we omit here. Thus we may conclude that the equivalence classes of $(\mathbb{Z}_2)^n$-colorings on P^n bijectively correspond to the equivariant diffeomorphism classes of small covers over P^n.

2.3. **Burnside Lemma and automorphism group of a prism.** The equivalence classes of $(\mathbb{Z}_2)^n$-colorings on P^n can be naturally considered as orbits of $\Lambda(P^n)$ under the action of $\mathrm{Aut}(\mathcal{F}(P^n))$. The famous Burnside Lemma (or Cauchy-Frobenius Lemma) is essential in the enumeration of the number of orbits.

Burnside Lemma. *Let G be a finite group acting on a set X. Then the number of orbits of X under the action of G equals to $\frac{1}{|G|}\sum_{g\in G}|X_g|$, where $X_g = \{x \in X | gx = x\}$.*

Burnside Lemma suggests that, in order to determine the number of the equivalence classes of $(\mathbb{Z}_2)^n$-colorings on P^n, we need to understand the structure of $\mathrm{Aut}(\mathcal{F}(P^n))$. As stated in Section 1, we shall particularly be concerned with the case in which P^n is a prism.

Let $P^3(m)$ be a m-sided prism (i.e., the product of a m-polygon $P^2(m)$ and the interval $[0,1]$, or two m-polygons joined by a belt of m squares).

Lemma 2.2. *When $m \neq 4$, the automorphism group $\mathrm{Aut}(\mathcal{F}(P^3(m)))$ is isomorphic to $\mathcal{D}_{2m} \times \mathbb{Z}_2$, where $\mathcal{D}_{2m}$ is the dihedral group of order $2m$. When $m = 4$, $\mathrm{Aut}(\mathcal{F}(P^3(m)))$ is isomorphic to the direct product $\mathcal{S}_4 \times \mathbb{Z}_2$, where $\mathcal{S}_4$ is the symmetric group of order 4.*

Proof. All sided facets of $P^3(m)$ are 4-polygons, and the top and bottom facets are two m-polygons. If $m \neq 4$, obviously there are automorphisms of $P^3(m)$ under which all sided facets admit an action of dihedral group of order $2m$, and there is also an automorphism of $P^3(m)$ such that the top and bottom facets is interchanged so they both admit an action of $\mathbb{Z}_2$. Since any one of all sided facets cannot be mapped to the top facet or bottom facet under the automorphisms of $P^3(m)$, we have that the automorphism group $\mathrm{Aut}(\mathcal{F}(P^3(m)))$ is just isomorphic to the direct product $\mathcal{D}_{2m} \times \mathbb{Z}_2$.

If $m = 4$, then all facets of $P^3(4)$ consists of six 4-polygons so $P^3(4)$ is a 3-cube. A 3-cube has the same automorphism group as an octahedron since a 3-cube is just the dual of an octahedron. Obviously, the automorphism group of an octahedron contains a symmetric group $\mathcal{S}_4$ of order 4 since there is exactly one such automorphism for each permutation of the four pairs of opposite sides of the octahedron. In addition, it is easy to see that an octahedron also admits a reflection automorphism which is different from any one of $\mathcal{S}_4$. Actually, such an automorphism reflects two opposite vertices of the octahedron. Thus, $\mathrm{Aut}(\mathcal{F}(P^3(4)))$ is isomorphic to the direct product $\mathcal{S}_4 \times \mathbb{Z}_2$. $\square$

Remark. It is not difficult to see that all automorphisms of $P^3(4)$ contain those automorphisms that combine a reflection and a rotation. Thus, $\mathrm{Aut}(\mathcal{F}(P^3(4)))$ has three versions of $\mathcal{D}_8 \times \mathbb{Z}_2$ as subgroups.

By s_1 and s_2 we denote the top and bottom facets of $P^3(m)$ respectively, and by $a_1, ..., a_m$ we denote all sided facets (i.e., 4-polygons) of $P^3(m)$ in their general order. Let x, y, z be three automorphisms of $\mathrm{Aut}(\mathcal{F}(P^3(m)))$ with the following properties respectively:

(1) $x(a_i) = a_{i+1} (i = 1, 2, ..., m-1), x(a_m) = a_1, x(s_j) = s_j, j = 1, 2;$
(2) $y(a_i) = a_{m+1-i} (i = 1, 2, ..., m), y(s_j) = s_j, j = 1, 2;$
(3) $z(a_i) = a_i (i = 1, 2, ..., m), z(s_1) = s_2, z(s_2) = s_1.$

Then, when $m \neq 4$, all automorphisms of $\mathrm{Aut}(\mathcal{F}(P^3(m)))$ can be written in a simple form as follows:

$$x^u y^v z^w,\ u \in \mathbb{Z}_m,\ v, w \in \mathbb{Z}_2 \tag{2.1}$$

with $x^m = y^2 = z^2 = 1$, $x^u y = y x^{m-u}$, and $x^u y^v z = z x^u y^v$.

3. Colorings on a prism

This section is devoted to calculating the number of $(\mathbb{Z}_2)^3$-colorings on a m-sided prism $P^3(m)$.

Theorem 3.1. *Let a, b, c be the functions from $\mathbb{N}$ to $\mathbb{N}$ with the following properties:*

1) $a(j) = 2a(j-1) + 8a(j-2)$ *with* $a(1) = 1, a(2) = 2;$

2) $b(j) = b(j-1) + 4b(j-2)$ *with* $b(1) = b(2) = 1$*;*

3) $c(j) = 2c(j-1) + 4c(j-2) - 6c(j-3) - 3c(j-4) + 4c(j-5)$ *with* $c(1) = c(2) = 1, c(3) = 3, c(4) = 7, c(5) = 17$*. Then the number of* $(\mathbb{Z}_2)^3$*-colorings on a* m*-sided prism* $P^3(m)$ *is*

$$|\Lambda(P^3(m)| = 168[a(m-1) + 2b(m-1) + c(m-1)].$$

Proof. $(\mathbb{Z}_2)^3$ contains seven nonzero elements (or seven colors) $e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3$ where e_1, e_2, e_3 form a standard basis of $(\mathbb{Z}_2)^3$. Set

$$A(m) = \{\lambda \in \Lambda(P^3(m)) \big| \lambda(s_1) = e_1, \lambda(a_1) = e_2, \lambda(a_2) = e_3\}.$$

Then, by Lemma 2.1, we have that $|\Lambda(P^3(m))| = |A(m)| \times |GL(3, \mathbb{Z}_2)| = 168|A(m)|$. Write

$$\begin{cases} A_1(m) = \{\lambda \in A(m) \big| \lambda(s_2) = e_1\} \\ A_2(m) = \{\lambda \in A(m) \big| \lambda(s_2) = e_1 + e_2\} \\ A_3(m) = \{\lambda \in A(m) \big| \lambda(s_2) = e_1 + e_3\} \\ A_4(m) = \{\lambda \in A(m) \big| \lambda(s_2) = e_1 + e_2 + e_3\}. \end{cases}$$

By the definition of $(\mathbb{Z}_2)^3$-colorings, it is easy to see that $|A(m)| = \sum_{i=1}^4 |A_i(m)|$. Then, our argument proceeds as follows.

Case 1. Calculation of $|A_1(m)|$.

Take a coloring λ in $A_1(m)$, by the definition of $(\mathbb{Z}_2)^3$-colorings, we easily see that $\lambda(a_m)$ has four possible values $e_3, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3$, and it is also so even if $\lambda(a_1) = e_1 + e_2$ since $\lambda(s_1) = \lambda(s_2) = e_1$. Set $A_1^0(m) = \{\lambda \in A_1(m) | \lambda(a_{m-1}) = e_2 \text{ or } e_1 + e_2\}$ and $A_1^1(m) = A_1(m) \setminus A_1^0(m)$. Take a coloring λ in $A_1^0(m)$. Then $\lambda(a_{m-1}) = e_2$ or $e_1 + e_2$, and so the possible values of $\lambda(a_{m-2})$ are $e_3, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3$. In this case, we see that the values of λ restricted to facets a_{m-1} and a_m have only eight possible choices. Thus, $|A_1^0(m)|$ is just eight times of $|A_1(m-2)|$. Take a coloring λ in $A_1^1(m)$. Then the possible values of $\lambda(a_{m-1})$ are $e_3, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3$. In this case, if we fix any one of four possible values of $\lambda(a_{m-1})$, then it is easy to see that $\lambda(a_m)$ has only two possible values. Thus, $|A_1^1(m)|$ is just two times of $|A_1(m-1)|$. Further, we have that

$$|A_1(m)| = 2|A_1(m-1)| + 8|A_1(m-2)|.$$

A direct observation shows that when $m = 2$, $|A_1(m)| = 1$, and when $m = 3$, $|A_1(m)| = 2$. Thus, we have that $|A_1(m)| = a(m-1)$.

Case 2. Calculation of $|A_2(m)|$.

Similarly to the case 1, set $A_2^0(m) = \{\lambda \in A_2(m) | \lambda(a_{m-1}) = e_2\}$ and $A_2^1(m) = A_2(m) \setminus A_2^0(m)$. Take a coloring λ in $A_2^0(m)$, we have that each of both $\lambda(a_{m-2})$ and $\lambda(a_m)$ has four possible values $e_3, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3$, so $|A_2^0(m)| = 4|A_2(m-2)|$; take a coloring λ in $A_2^1(m)$, we then have that $\lambda(a_{m-1})$ has four possible values $e_3, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3$ but $\lambda(a_m)$ has only one possible value whichever of four possible values of $\lambda(a_{m-1})$ is chosen, so $|A_2^1(m)| = |A_2(m-1)|$. Also, we easily see that $|A_2(2)| = |A_2(3)| = 1$. Thus, $|A_2(m)| = b(m-1)$.

Case 3. Calculation of $|A_3(m)|$.

If we interchange e_2 and e_3 in this case, then the problem is reduced to the case 2, so $|A_3(m)| = b(m-1)$.

Case 4. Calculation of $|A_4(m)|$.

In this case, given a coloring $\lambda \in A_4(m)$, we have that for any sided facet a_i, $\lambda(a_i)$ cannot be e_1 and $e_1+e_2+e_3$. Note that $\lambda(a_m)$ is equal to e_3 or e_2+e_3 since $\lambda(s_2) = e_1+e_2+e_3$. Set $A_4^0(m) = \{\lambda \in A_4(m) | \lambda(a_{m-1}) = e_2\}$, $A_4^1(m) = \{\lambda \in A_4(m) | \lambda(a_{m-1}) = e_3 \text{ or } e_2+e_3\}$, and $A_4^2(m) = \{\lambda \in A_4(m) | \lambda(a_{m-1}) = e_1+e_2 \text{ or } e_1+e_3\}$. Then $|A_4(m)| = |A_4^0(m)| + |A_4^1(m)| + |A_4^2(m)|$. An easy argument shows that $|A_4^0(m)| = 2|A_4(m-2)|$ and $|A_4^1(m)| = |A_4(m-1)|$, so

$$|A_4(m)| = |A_4(m-1)| + 2|A_4(m-2)| + |A_4^2(m)|. \tag{3.1}$$

Set $B(m) = \{\lambda \in A_4^2(m) | \lambda(a_{m-2}) = e_2 + e_3\}$. Then it is easy to see that

$$|A_4^2(m)| = |A_4^2(m-1)| + |B(m)| \tag{3.2}$$

and

$$|B(m)| = 2|A_4(m-4)| + 2|A_4(m-5)| + |B(m-2)| + 2|A_4^2(m-2)|. \tag{3.3}$$

Combining equations (3.1), (3.2) and (3.3), we obtain

$$|A_4(m)| = 2|A_4(m-1)| + 4|A_4(m-2)| - 6|A_4(m-3)| - 3|A_4(m-4)| + 4|A_4(m-5)|.$$

A direct observation gives that $|A_4(2)| = |A_4(3)| = 1, |A_4(4)| = 3, |A_4(5)| = 7$, and $|A_4(6)| = 17$. Thus, we have $|A_4(m)| = c(m-1)$. $\square$

Note that an easy argument shows $a(j) = \frac{1}{6}[4^j - (-1)^j 2^j]$. Also, using a different argument, we can obtain another expression of $c(j)$, i.e., $c(j) = 2c(j-1) + 3c(j-2) - 4c(j-3) + (-1)^j 2$.

4. The number of equivariant diffeomorphism classes

Let $\varphi(n)$ be the value at $n \in \mathbb{N}$ of the Euler's totient function, which is the number of positive integers that are both less than n and coprime to n. Specifically, if we write $n = p_1^{l_1} \cdots p_r^{l_r}$ where the p_i are distinct primes, then $\varphi(n) = (p_1^{l_1} - p_1^{l_1-1}) \cdots (p_r^{l_r} - p_r^{l_r-1})$. Note that $\varphi(1) = 1$. Let $\nu(m)$ denote the number of $(\mathbb{Z}_2)^3$-colorings on $P^3(m)$ with its top and bottom facets having the same color, so $\nu(m) = 168a(m-1)$. Let $E(m)$ denote the number of equivariant diffeomorphism classes of small covers over $P^3(m)$, which is just the number of the equivalence classes of all $(\mathbb{Z}_2)^3$-colorings on $P^3(m)$.

Theorem 4.1. *The number $E(m)$ is equal to*

$$\begin{cases} \frac{1}{4m}\left\{ \sum_{k>1, k|m} \varphi(\frac{m}{k})[|\Lambda(P^3(k))| + \nu(k)] + 21m\rho_1(m) + 42m\rho_2(m)\right\} & \textit{if } m \neq 4 \\ 259 & \textit{if } m = 4 \end{cases}$$

where $\rho_1(m)$ is defined recursively as follows

$$\rho_1(m) = \begin{cases} 0 & m \text{ is odd} \\ 5 & m = 0 \\ 12 & m = 2 \\ \rho_1(m-2) + 4\rho_1(m-4) & \text{otherwise} \end{cases}$$

and

$$\rho_2(m) = \begin{cases} 0 & m \text{ is odd} \\ 2^m & m \text{ is even.} \end{cases}$$

Proof. From Burnside Lemma and Lemma 2.2, we have that

$$E(m) = \begin{cases} \frac{1}{4m} \sum_{g \in \mathrm{Aut}(\mathcal{F}(P^3(m)))} |\Lambda_g| & \text{if } m \neq 4 \\ \frac{1}{48} \sum_{g \in \mathrm{Aut}(\mathcal{F}(P^3(4)))} |\Lambda_g| & \text{if } m = 4 \end{cases}$$

where $\Lambda_g = \{\lambda \in \Lambda(P^3(m)) | \lambda = \lambda \circ g\}$.

If $m \neq 4$, then by (2.1) each automorphism g of $\mathrm{Aut}(\mathcal{F}(P^3(m))$ can be written as $x^u y^v z^w$, and the argument is divided into the following cases.

Case 1: $g = x^u$.

In this case, g is just an automorphism which rotates all sided facets of $P^3(m)$. Let $k = \gcd(u, m)$. Then all sided facets of $P^3(m)$ are divided into k orbits under the action of g, and each orbit contains $\frac{m}{k}$ facets. Thus, each $(\mathbb{Z}_2)^3$-coloring of Λ_g gives the same coloring on all $\frac{m}{k}$ facets of each orbit. This means that if $k \neq 1$, $|\Lambda_g|$ exactly equals to the number of $(\mathbb{Z}_2)^3$-colorings on a k-sided prism, so $|\Lambda_g| = |\Lambda(P^3(k))|$. If $k = 1$, then all sided facets have the same coloring, which is impossible by the definition of $(\mathbb{Z}_2)^3$-colorings. On the other hand, for every $k > 1$, there are exactly $\varphi(\frac{m}{k})$ automorphisms of the form x^u, each of which divides all sided facets of $P^3(m)$ into k orbits. Thus, when g is of the form x^u,

$$\sum_{g = x^u} |\Lambda_g| = \sum_{k > 1, k | m} \varphi(\frac{m}{k}) |\Lambda(P^3(k))|.$$

Case 2: $g = x^u z$.

Each such automorphism $g = x^u z$ gives an interchange between top and bottom facets, so each $(\mathbb{Z}_2)^3$-coloring of Λ_g gives the same coloring on top and bottom facets. Combining the argument of the case 1, when g is of the form $x^u z$,

$$\sum_{g = x^u z} |\Lambda_g| = \sum_{k > 1, k | m} \varphi(\frac{m}{k}) \nu(k).$$

Case 3: $g = x^u y$ or $x^u yz$ with m odd.

In this case, since m is odd, each automorphism always reflects at least two neighborly sided facets, so that the two neighborly sided facets have the same coloring. But this

contradicts the definition of $(\mathbb{Z}_2)^3$-colorings. Thus, for each such an automorphism g, Λ_g is empty.

Case 4: $g = x^u y$ or $x^u yz$ with u even and m even.

Let $l = \frac{m-u-2}{2}$. Then it is easy to see that such an automorphism in this case gives an interchange between two sided facets a_l and a_{l+1}, so both facets a_l and a_{l+1} have the same coloring, exactly as in the case 3. Thus, in this case Λ_g is empty, too.

Case 5: $g = x^u y$ with u odd and m even.

Since each automorphism $g = x^u y$ contains a reflection y as its factor and u is odd, it becomes a reflection along a plane through the center of some sided facet. Thus, each coloring λ of Λ_g is equivalent to coloring only $\frac{m}{2}+1$ neighborly sided facets and top and bottom facets of $P^3(m)$. We shall show that for each $g = x^u y$, the number of all colorings in Λ_g is just

$$|\Lambda_g| = 42\rho_1(m) + 42\rho_2(m) \tag{4.1}$$

where $\rho_1(m)$ and $\rho_2(m)$ are stated as in Theorem 4.1. It is easy to see that there are exactly $\frac{m}{2}$ such automorphisms $g = x^u y$ since m is even and u is odd, so

$$\sum_{g=x^u y} |\Lambda_g| = 21m\rho_1(m) + 21m\rho_2(m).$$

Now let us show the equality (4.1) as follows.

Actually, the argument method of Case 1 of Theorem 3.1 can still be carried out here. Also, it suffices to consider the case $g = x^{m-1}y$ (i.e., $g = yx$) since other cases have no difference essentially. Set $X_1(m) = \{\lambda \in \Lambda_g | \lambda(s_1) \neq \lambda(s_2)\}$ and $X_2(m) = \Lambda_g \setminus X_1(m)$. Then, by $X_1^0(m)$ we denote the set $\{\lambda \in X_1(m) | \lambda(a_m) \neq \lambda(s_1) + \lambda(a_2), \lambda(a_m) \neq \lambda(s_2) + \lambda(a_2), \lambda(a_m) \neq \lambda(a_2)\}$, and by $X_1^1(m)$ denote $X_1(m) \setminus X_1^0(m)$. Similarly to the argument of Case 1 of Theorem 3.1, we have that $|X_1^0(m)| = |X_1(m-2)|$ and $|X_1^1(m)| = 4|X_1(m-4)|$ with initial values $X_1(0) = 5 \times 6 \times 7$ and $X_1(2) = 12 \times 6 \times 7$. Thus, $|X_1(m)| = 42\rho_1(m)$. For $X_2(m)$, in a similar way we may obtain $|X_2(m)| = 4|X_2(m-2)|$ with $X_2(0) = 42$, which is exactly $42\rho_2(m)$.

Case 6: $g = x^u yz$ with u odd and m even.

This case is the same as $X_2(m)$ of Case 5. Thus, $\sum_{g=x^u yz} |\Lambda_g| = 21m\rho_2(m)$.

Combining Cases 1-6, we complete the proof in the case $m \neq 4$.

If $m = 4$, then $P^3(4)$ is a 3-cube. The automorphism group of a 3-cube has order 48, and contains $\mathcal{D}_8 \times \mathbb{Z}_2$ of order 16 as a subgroup. As shown above, actually we have determined the case of the action of $\mathcal{D}_8 \times \mathbb{Z}_2$ on $P^3(4)$. However, each of other 32 automorphisms of $P^3(4)$ has no fixed coloring in $\Lambda(P^3(4))$ since it maps top facet (or bottom facet) to a sided facet. Thus

$$E(4) = \frac{1}{48}\{\sum_{k=2,4} \varphi(\frac{4}{k})[|\Lambda(P^3(k))| + \nu(k)] + 84\rho_1(4) + 168\rho_2(4)\} = 259.$$

$\square$

Remark. We may write a computer program to calculate $E(m)$ by the formula in Theorem 4.1. The first 8 numbers are stated as follows.

m	3	4	5	6	7	8	9	10
$E(m)$	98	259	882	4200	9114	35406	107086	394632

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